Unit – I : Laplace Transform

Definition

Let F(t) be a function of t defined for all $t \geq 0.$ Then the Laplace transform of F(t), denoted by $L\{F(t)\}$, is defined by

$$\boxed{L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt}$$

Provided that the integral exists, 'p' is a parameter which may be real or complex.

L{F(t)} is said to exist if the above integral converges for some value of p otherwise not.

The function f(p) is called the Laplace transform or the image of the object function F(t).

Remark 1. Some authors use the letter s for the parameter instead of p. Therefore, we may also write

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s).$$

Remark 2. In general, we will denote the object function by a capital letter and its transform by the same letter in lower case. But other notations that distinguish between functions and their transforms are sometimes preferable

e.g., $L\{F(t)\} = \phi(p)$ or $L\{y(t)\} = \overline{y}(p)$ or $L\{f(t)\} = \overline{f}(p)$ etc.

Linearity property

If c_1, c_2 are constants and f, g are functions of t, then

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\}$$

By definition, $L\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-pt} \{c_1 f(t) + c_2 g(t)\}\, dt$

$$= c_1 \int_0^\infty e^{-pt} f(t)\, dt + c_2 \int_0^\infty e^{-pt} g(t)\, dt = c_1 L\{f(t)\} + c_2 L\{g(t)\}$$

The result can easily be generalized.

Laplace transform of some elementary functions

(1) $L\{1\}=\dfrac{1}{p},\ p>0$

$$L\{1\}=\int_0^\infty e^{-pt}.1dt=\left[-\frac{e^{-pt}}{p}\right]_0^\infty=\frac{1}{p},\ \text{if}\ p>0$$

(2) $L\{t^n\}=\dfrac{n!}{p^{n+1}}$, where n is a positive integer.

$$L\{t^n\}=\int_0^\infty e^{-pt}.t^n\,dt=\int_0^\infty e^{-x}\left(\frac{x}{p}\right)^n\frac{dx}{p},\ \text{on putting}\ pt=x$$

$$=\frac{1}{p^{n+1}}\int_0^\infty x^n e^{-x}dx=\frac{\Gamma(n+1)}{p^{n+1}}\ \text{provided that}\ p>0\ \text{and}\ n+1>0\,i.e., n>-1.$$

If n is a positive integer, $\Gamma(n+1)=n!$

$\therefore$ $$L\{t^n\}=\frac{n!}{p^{n+1}}.$$

Note. For $n=1$, $L(t)=\dfrac{1}{p^2}$.

(3) $L\{e^{at}\}=\dfrac{1}{p-a}, p>a$

$$L\{e^{at}\}=\int_0^\infty e^{-pt}.e^{at}dt=\int_0^\infty e^{-(p-a)t}dt=\left[-\frac{e^{(p-a)t}}{p-a}\right]_0^\infty=\frac{1}{p-a},\ \text{if}\ p>a.$$

(4) $L\{\sin at\}=\dfrac{a}{p^2+a^2}, p>a$

$$L\{\sin at\}=\int_0^\infty e^{-pt}\sin at\,dt=\left[-\frac{e^{-pt}}{p^2+a^2}(-p\sin at-a\cos at)\right]_0^\infty=\frac{a}{p^2+a^2}.$$

(5) $L\{\cos at\}=\dfrac{p}{p^2+a^2}, p>0$

$$L\{\cos at\} = \int_0^\infty e^{-pt}.\cos at\,dt = \left[-\frac{e^{-pt}}{p^2+a^2}(-p\cos at + a\sin at)\right]_0^\infty = \frac{p}{p^2+a^2}.$$

(6) $L\{\sinh at\} = \dfrac{a}{p^2-a^2},\ p>|0|.$

$$L\{\sinh at\} = \int_0^\infty e^{-pt}\sinh\ at\,dt = \int_0^\infty e^{-pt}.\left[\frac{e^{at}-e^{-at}}{2}\right]dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(p-a)t}dt - \int_0^\infty e^{-(p+a)t}dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{p-a} - \frac{1}{p+a}\right] = \frac{a}{p^2-a^2},\ \text{for}\ \ p>|a|.$$

Note. We can also prove it by using linearity property.

$$\text{Thus}\ \ L\{\sinh at\} = L\left\{\frac{1}{2}(e^{at}-e^{-at})\right\} = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left(\frac{1}{p-a}\right) - \frac{1}{2}\left(\frac{1}{p+a}\right) = \frac{a}{p^2-a^2}$$

(7) $L\{\cosh at\} = \dfrac{p}{p^2-a^2},\ p>|a|.$

$$L\{\cosh at\} = L\left\{\frac{1}{2}(e^{at}+e^{-at})\right\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\left(\frac{1}{p-a}\right) + \frac{1}{2}\left(\frac{1}{p+a}\right) = \frac{p}{p^2-a^2},\ \text{for}\ \ p>|a|.$$

For ready reference, the Laplace transforms of various elementary function have been listed in the following table :

F(t)	$L[F(t)] = f(p)$
1	$\dfrac{1}{p},\ p>0$

t	$\frac{1}{p^2}, p>0$
t^n, n is a positive integer	$\frac{n!}{p^{n+1}}, p>0$
$t^n,\ n>-1$	$\frac{\Gamma(n+1)}{p^{n+1}}, p>0$
e^{at}	$\frac{1}{p-a}, p>a$
e^{-at}	$\frac{1}{p+a}$
$\sin at$	$\frac{a}{p^2+a^2}, p>0$
$\cos at$	$\frac{p}{p^2+a^2}, p>0$
$\sinh at$	$\frac{a}{p^2-a^2}, p>\|a\|$
$\cosh at$	$\frac{p}{p^2-a^2}, p>\|a\|$

TRANSFORMS OF DISCONTINUOUS FUNCTIONS

The Laplace transform of F(t) will exist even if the object function F(t) is discontinuous, provided the integral in the definition of $L\{F(t)\}$ exists.

First Translation Property Or First Shifting Property

If $L\{F(t)\} = f(p)$ then $L\{e^{at}F(t)\} = f(p-a)$.

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-pt}.e^{at}F(t)dt \qquad | \text{ By definition}$$

$$= \int_0^\infty e^{-(p-a)t}F(t)dt = f(p-a).$$

Remark 1. $L\{e^{-at}F(t)\} = f(p+a)$.

Remark 2. $L\{e^{at}F(bt)\} = \frac{1}{b} f\left(\frac{p-a}{b}\right)$.

Applying this property to the elementary functions of Art. 3.4, we get the following useful results.

(1) $L\{e^{at}t^n\} = \dfrac{n!}{(p-a)^{n+1}}$; n is a positive integer.

(2) $L\{e^{at}\sin bt\} = \dfrac{b}{(p-a)^2+b^2}$

(3) $L\{e^{at}\cos bt\} = \dfrac{p-a}{(p-a)^2+b^2}$

(4) $L\{e^{at}\sinh bt\} = \dfrac{b}{(p-a)^2-b^2}$

(5) $L\{e^{at}\cosh bt\} = \dfrac{p-a}{(p-a)^2-b^2}$

3.03 Second Translation Property Or Haviside's Shifting Theorem

If $L[F(t)] = f(p)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

Then, $L\{G(t)\} = e^{-ap} f(p)$.

$$L\{G(t)\} = \int_0^\infty e^{-pt}.G(t)dt = \int_0^\infty e^{-pt}G(t)dt + \int_a^\infty e^{-pt}G(t)dt$$

$$= 0 + \int_a^\infty e^{-pt}.F(t-a)dt = \int_a^\infty e^{-pt}.F(t-a)dt$$

Put $t-a = u \Rightarrow dt = du$

$$= \int_a^\infty e^{-p(u+a)}F(u)du = e^{-pa}\int_a^\infty e^{-pu}.F(u)du$$

$$= e^{-ap}\int_0^\infty e^{-pt}F(t)dt = e^{-ap} f(p).$$

3.04 Change of Scale Property :

If $L[F(t)] = f(p)$ then $L\{F(at)\} = \dfrac{1}{a} f\left(\dfrac{p}{a}\right)$.

$$L\{F(at)\} = \int_0^\infty e^{-pt}F(at)dt$$

Put $at = u \Rightarrow dt = \dfrac{du}{a}$

$$= \int_0^\infty e^{-p\frac{u}{a}}F(u)\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)u}F(u)du$$

$$=\frac{1}{a}\int_0^{\infty} e^{-\left(\frac{p}{a}\right)t} F(t)dt=\frac{1}{a} f\left(\frac{p}{a}\right).$$

Q.1 Find the Laplace transform of $e^{-3t}(\cos 4t+3\sin 4t)$.

Sol. $L(\cos 4t+3\sin 4t)=L(\cos 4t)+3L(\sin 4t)$

$$=\frac{p}{p^2+16}+\frac{12}{p^2+16}=\frac{p+12}{p^2+16}$$

$\therefore$ $L\{e^{-3t}(\cos 4t+3\sin 4t)\}=\frac{(P+3)+12}{(P+3)^2+16}$ | Using first shifting property

$$=\frac{P+15}{P^2+6p+25}.$$

Q.2 Find the Laplace transform of

(i) $F(t)=\begin{cases}\cos t, & 0<t<\pi \\ 0, & t>\pi\end{cases}$ (ii) $F(t)=\begin{cases}1\ , & 0\le t<1 \\ t\ , & 1\le t<2 \\ t^2, & 2\le t<\infty\end{cases}$

Sol. (i) $L\{F(t)\}=\int_0^{\infty} e^{-pt}.F(t)dt=\int_0^{\pi} e^{-pt}.\cos t dt+\int_{\pi}^{\infty} e^{-pt}.0\,dt$

$$=\left[\frac{e^{-pt}}{p^2+1}(-p\cos t+\sin t)\right]_0^{\pi}=\left[\frac{e^{-p\pi}}{p^2+1}p-\frac{1}{p^2+1}(-p)\right]$$

$$=\frac{p(1+e^{-p\pi})}{p^2+1}$$

(ii) $L\{F(t)\}=\int_0^{\infty} e^{-pt}.F(t)dt=\int_0^{1} e^{-pt}dt+\int_1^{2} t\,e^{-pt}\,dt+\int_2^{\infty} t^2e^{-pt}dt$

$$=\left(\frac{e^{-pt}}{-p}\right)_0^1+\left(t\frac{e^{-pt}}{-p}-\frac{e^{-pt}}{p^2}\right)_1^2+\left(t^2\frac{e^{-pt}}{-p}\right)_2^{\infty}-\int_2^{\infty} 2t.\frac{e^{-pt}}{-p}dt$$

$$=\left(\frac{1-e^{-p}}{p}\right)+\left(\frac{-2}{p}e^{-2p}-\frac{e^{-2p}}{p^2}\right)-\left(\frac{e^{-p}}{-p}\frac{e^{-p}}{p^2}\right)+\frac{4}{p}e^{-2p}+\frac{2}{p}\int_2^{\infty} t\,e^{-pt}dt$$

$$=\frac{1}{p}+\frac{2}{p}e^{-2p}+\frac{e^{-p}}{p^2}-\frac{e^{-2p}}{p^2}+\frac{2}{p}\left[\left(t\frac{e^{-pt}}{-p}\right)_2^{\infty}-\int_2^{\infty}1.\frac{e^{-pt}}{-p}dt\right]$$

$$=\frac{1}{p}+\frac{2}{p}e^{-2p}+\frac{e^{-p}}{p^2}-\frac{e^{-2p}}{p^2}+\frac{2}{p}\left[\frac{2}{p}e^{-2p}+\frac{1}{p}\left(\frac{e^{-pt}}{-p}\right)_2^{\infty}\right]$$

$$=\frac{1}{p}+\frac{2}{p}e^{-2p}+\frac{e^{-p}}{p^2}+\frac{3}{p^2}e^{-2p}+\frac{2}{p^3}e^{-2p}.$$

Existence theorem

If $F(t)$ is sectionallly continous for $t \geq 0$ and is of exponential order b, then

$$L\{F(t)\}=f(p) \text{ exists for } p>b.$$

In other words, if $F(t)$ is function of class A, L $\{F(t)\}$ exists.

$$\int_0^{\infty} e^{-pt}F(t)dt=\int_0^{t_0} e^{-pt}.F(t)dt+\int_{t_0}^{\infty} e^{-pt}.F(t)dt=I_1+I_2 \qquad \text{(say)}$$

I_1 exists since $F(t)$ is sectionally continuous in every finite interval $0 \leq t \leq t_0$.

$$|I_2| \leq \int_{t_0}^{\infty}|e^{-pt}.F(t)|\,dt \leq \int_0^{\infty} |e^{-pt}.F(t)|\,dt$$

$$\leq \int_0^{\infty} Me^{bt}dt \qquad \text{as } F(t) \text{ is of exponential order b}$$

$$\leq \int_0^{\infty} e^{-(p-b)t}.M\,dt=\frac{M}{p-b}.$$

Thus the Laplace transform exists for $p>b$.

Note- The conditions of the theorem are sufficient but not necessary for the existence of Laplace transform.

3.05 Laplace Transform of Derivative :

Theorem 1 $L\{F'(t)\}=pL\{F(t)\}-F(0)=p[f(p)-F(0).$

INITIAL – VALUE THEOREM

If $F(t)$ is continuous for all $t \geq 0$ and is of exponential order as $t \to \infty$ and if $F'(t)$ is of class A then

$$\lim_{t \to 0} F(t) = \lim_{p \to \infty} pL\{F(t)\}$$

FINAL – VALUE THEOREM

If $F(t)$ is continuous for all $t \geq 0$ and is of exponential order as $t \to \infty$ and if $F'(t)$ is of class A then

$$\lim_{t \to \infty} F(t) = \lim_{p \to 0} pL\{F(t)\}.$$

3.06 Laplace Transform of Integrals

If $L\{F(t)\} = f(p)$, then $L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{p} f(p)$

Let $G(t) = \int_0^t F(t)\,dt,$ then

$$G'(t) = F(t) \text{ and } G(0) = 0$$

Taking Laplace transform, we get

$$L\{G'(t)\} = pL\{G(t)\} - G(0) = pL\{G(t)\}$$

$\therefore$ $L\{G(t)\} = \frac{1}{p} L\{G'(t)\} = \frac{1}{p} L\{F(t)\} = \frac{1}{p} f(p)$

i.e., $L\left\{\int_0^t F(t)\,dt\right\} = \frac{1}{p} f(p)$

3.07 Multiplications By t^n

If $L\{F(t)\} = f(p)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n}[f(p)]$, where $n = 1, 2, 3, ...$

Division by t

If $\quad L\{F(t)\} = f(p),$ then $L\left\{\frac{1}{t}F(t)\right\} = \int_p^\infty f(p)\,dp$ provided the integral exists.

We have $\quad f(p) = \int_0^\infty e^{-pt}F(t)dt$

Integrating both sides w.r.t. p from p to ∞, we have

$$\int_p^\infty f(p)dp = \int_p^\infty \left[\int_0^\infty e^{-pt}F(t)\,dt\right]dp$$

Since p and t are independent, changing the order of integration on the right-hand side, we have

$$\int_p^\infty f(p)dp = \int_0^\infty \left[\int_p^\infty e^{-pt}\,dp\right]F(t)dt$$

$$= \int_0^\infty \left[\frac{e^{-pt}}{-t}\right]_p^\infty F(t)dt = \int_0^\infty e^{-pt}\frac{F(t)}{t}dt = L\left\{\frac{1}{t}F(t)\right\}.$$

Q.3 Find the Laplace transform of

(i) t^3e^{-3t} (ii) $t\sin^2 3t$

Sol. (i) $\quad L(t^3) = \dfrac{6}{p^4}$

$\therefore \quad L(e^{-3t}t^3) = \dfrac{6}{(p+3)^4}$

(ii) $\quad \sin^2 3t = \dfrac{1-\cos 6t}{2}$

$\therefore \quad L(\sin^2 3t) = \dfrac{1}{2}[L(1) - L(\cos 6t)] = \dfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{p}{p^2+36}\right) = \dfrac{18}{p(p^2+36)}$

$\therefore \quad L(t\sin^2 3t) = \dfrac{d}{dp}\left[\dfrac{18}{p(p^2+36)}\right]$

$$= (-18)(-1)(p^3+36p)^{-2}(3p^2+36) = \frac{54(p^2+12)}{p^2(p^2+36)^2}.$$

Q.4 Evalulate :

(i) $\int_0^\infty t^3 e^{-t} \sin t \, dt$ (ii) $\int_0^\infty \dfrac{e^{-t}\sin^2 t}{t} dt$

Sol. $L(\sin t) = \dfrac{1}{p^2+1}$

$$L(t^3 \sin t) = (-1)^3 \frac{d^3}{dp^3}\left(\frac{1}{p^2+1}\right)$$

$$= \frac{d^2}{dp^2}\left[\frac{2p}{(p^2+1)^2}\right] = \frac{d}{dp}\left[\frac{2(1-3p^2)}{(p^2+1)^3}\right]$$

$$L(t^3 \sin t) = \frac{24p(p^2-1)}{(p^2+1)^4}$$

By definition, $\int_0^\infty e^{-pt}.t^3 \sin t \, dt = \dfrac{24p(p^2-1)}{(p^2+1)^4}$

Put $P = 1$

$$\int_0^\infty e^{-t}\, t^3 \sin t \, dt = 0 .$$

(ii) $L(\sin^2 t) = \dfrac{1}{2} L(1-\cos 2t) = \dfrac{1}{2}\left(\dfrac{1}{2} - \dfrac{p}{p^2+4}\right)$

$$L\left(\frac{\sin^2 t}{t}\right) = \frac{1}{2}\int_p^\infty \left(\frac{1}{p} - \frac{p}{p^2+4}\right) dp = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$$

By definition, $\int_0^\infty e^{-pt}.\dfrac{\sin^2 t}{t} dt = \dfrac{1}{4}\log\left(\dfrac{p^2+4}{p^2}\right)$

Put $p = 1$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5 .$$

Q.5 Find the Laplace transform of

(i) $\int_0^t e^{-t}\cos t\,dt$ (ii) $\int_0^t \frac{\sin t}{t}\,dt$ (iii) $\int_0^t e^t\frac{\sin t}{t}\,dt$

Sol. (i) $L(\cos t) = \dfrac{p}{p^2+1}$

$$L(e^{-t}\cos t) = \frac{p+1}{(p+1)^2+1} = f(p) \qquad \text{(say)}$$

$\therefore$ $$L\left(\int_0^t e^{-t}\cos t\,dt\right) = \frac{1}{p}f(p) = \frac{p+1}{p(p^2+2p+2)}.$$

(ii) $L(\sin t) = \dfrac{1}{p^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_p^\infty \frac{1}{p^2+1}\,dp = \frac{\pi}{2} - \tan^{-1}p = \cot^{-1}p$$

$\therefore$ $$L\left(\int_0^t \frac{\sin t}{t}\,dt\right) = \frac{1}{p}\cot^{-1}p.$$

(iii) $L\left(\dfrac{\sin t}{t}\right) = \cot^{-1}p$ | as done in part (ii)

$\therefore$ $$L\left(e^t\frac{\sin t}{t}\right) = \cot^{-1}(p-1)$$

$\therefore$ $$L\left(\int_0^t e^t\frac{\sin t}{t}\,dt\right) = \frac{1}{p}\cot^{-1}(p-1).$$

3.08 Unit step Function (or Heaviside's Unit Step Function)

The unit step function $u(t-a)$ is defined as

$$u(t-a) = \begin{cases} 0, & for\ t < a \\ 1, & for\ t \ge a \end{cases}, \text{ where } a \ge 0.$$

As a particular case, $u(t) = \begin{cases} 0, & for\ t < 0 \\ 1, & for\ t \geq 0 \end{cases}$ fig

The product $F(t).u(t-a) = \begin{cases} 0, & for\ t < a \\ F(t), & for\ t \geq a \end{cases}$

The function $F(t-a).u(t-a)$ represents the graph of $F(t)$ shifted through a distance 'a' to the right.

Laplace Transform of Unit Step Function

$$L\{u(t-a)\} = \int_0^\infty e^{-pt} u(t-a)dt$$

$$= \int_0^a e^{-pt}.0\,dt + \int_a^\infty e^{-pt}.1dt = 0 + \left[\frac{e^{-pt}}{-p}\right]_a^\infty = \frac{1}{p}e^{-ap}$$

In particular, $L\{u(t)\} = \frac{1}{p}$.

Laplace transform of Unit Impulse Function

If $f(t)$ is a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\delta_\varepsilon(t-a)dt = \int_a^{a+\varepsilon} f(t).\frac{1}{\varepsilon}dt$$

$$= (a+\varepsilon-a)f(c).\frac{1}{\varepsilon} = f(c), \text{ where } a < c < a+\varepsilon$$

(by mean value theorem for integrals)

As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\delta(t-a)dt = f(a)$.

Cor.1. $L\{\delta(t-a)\} = \int_0^\infty e^{-pt}\,\delta(t-a)dt = e^{-pa}$

Cor.2. $L\{\delta(t)\} = e^0 = 1.$

3.08 Periodic Functions

If $f(t)$ is a periodic function with period T, i.e., $f(t+T) = f(t)$, then

$$L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\, dt.$$

Q.6 Find the Laplace transform of the following periodic functions:

(i) $f(t) = t/T$, for $0 < t < T$ (saw-tooth wave of period T)

(ii) $f(t) = \sin\left(\frac{\pi t}{a}\right)$ for $0 < t < a$. (Rectified sine wave of period a)

Sol. (i) Heve, $L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t) dt = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} . \frac{t}{T} dt$

$$= \frac{1}{T(1-e^{-pT})}\left[\left(\frac{te^{-pt}}{-p}\right)_0^T - \int_0^T 1.\frac{e^{-pt}}{-p} dt\right]$$

$$= \frac{1}{1-e^{-pT}}\left[\frac{e^{-pT}}{p} + \frac{1-e^{-pT}}{p^2 T}\right] = \frac{1}{p^2 T} - \frac{e^{-pT}}{p(1-e^{pT})}$$

(ii) $$L\{f(t)\} = \frac{1}{1-e^{-ap}} \int_0^a e^{-pt}.\sin\left(\frac{\pi t}{a}\right) dt$$

$$I = \int_0^a e^{-pt}.\sin\left(\frac{\pi t}{a}\right) dt$$

$$= \left[\frac{e^{-pt}}{p^2 + \frac{\pi^2}{a^2}}\left(-p\sin\frac{\pi t}{a} - \frac{\pi}{a}\cos\frac{\pi t}{a}\right)\right]_0^a$$

$$= \left[\frac{e^{-ap}}{p^2 + \frac{\pi^2}{a^2}}\left(\frac{\pi}{a}\right)\right] - \left[\frac{1}{p^2 + \frac{\pi^2}{a^2}}.\left(\frac{\pi}{a}\right)\right] = \frac{(1+e^{-ap})a\pi}{a^2 p^2 + \pi^2}$$

$\therefore$ From (1), $L\{f(t)\} = \frac{(1+e^{-ap})a\pi}{a^2 p^2 + \pi^2}$

$$=\left(\frac{e^{ap/2}+e^{-ap/2}}{e^{ap/2}-e^{-ap/2}}\right)\left(\frac{a\pi}{a^2p^2+\pi^2}\right)=\frac{a\pi\coth\frac{ap}{2}}{a^2p^2+\pi^2}$$

3.09 Inverse Laplace transform

If $L\{F(t)\}=f(p)$, then $F(t)$ is called the inverse laplace transform of $f(p)$ and is denoted by

$$L^{-1}\{f(p)\}=F(t)$$

Here L^{-1} denotes the inverse Laplace transform operator.

e.g., Since $L\{e^{5t}\}=\frac{1}{p-5}$ $\therefore$ $L^{-1}\left\{\frac{1}{p-5}\right\}=e^{5t}$

The inverse Laplace transforms given below follow at once from the results of Laplace transforms given earlier :

(1) $L^{-1}\left\{\frac{1}{p}\right\}=1$

(2) $L^{-1}\left\{\frac{1}{p-a}\right\}=e^{at}$

(3) $L^{-1}\left\{\frac{1}{p^n}\right\}=\frac{t^{n-1}}{(n-1)!}$ if n is a positive ingeger. $\left[otherwise=\frac{t^{n-1}}{\Gamma(n)}\right]$

(4) $L^{-1}\left\{\frac{1}{(p-a)^n}\right\}=e^{at}\,\frac{t^{n-1}}{(n-1)!}$

(5) $L^{-1}\left\{\frac{1}{(p^2+a^2)}\right\}=\frac{1}{a}\sin at$

(6) $L^{-1}\left\{\frac{1}{(p^2+a^2)}\right\}=\cos at$

(7) $L^{-1}\left\{\frac{1}{(p^2-a^2)}\right\}=\frac{1}{a}\sinh at$

(8) $L^{-1}\left\{\frac{p}{(p^2-a^2)}\right\}=\cosh at$.

All the above results must be remembered.

Q.7 Evaluate :

(i) $L^{-1}\left(\frac{e^{-2p}}{p^2}\right)$ (ii) $L^{-1}\left(\frac{e^{-p}-3e^{-3p}}{p^2}\right)$ (iii) $L^{-1}\left(\frac{pe^{-ap}}{p^2-\omega^2}\right);a>0$

Sol.(i) We have

$$L^{-1}\left(\frac{1}{p^2}\right) = t = F(t)$$

$$\therefore \qquad L^{-1}\left(e^{-2p}.\frac{1}{p^2}\right) = \begin{cases} t-2 & , \quad t>2 \\ 0 & , \quad t<2 \end{cases} = (t-2)\,u\,(t-2).$$

|By second shifting theorem

(ii) $$L^{-1}\left(e^{-2p}.\frac{1}{p^2}\right) = \begin{cases} t-1 & , \quad t>1 \\ 0 & , \quad t<1 \end{cases} = (t-1)\,u\,(t-1)$$

$$L^{-1}\left(e^{-3p}.\frac{1}{p^2}\right) = \begin{cases} t-3 & , \quad t>3 \\ 0 & , \quad t<3 \end{cases} = (t-3)\,u\,(t-3)$$

Hence $$L^{-1}\left(\frac{e^{-p}-3e^{-3p}}{p^2}\right) = (t-1)\,u\,(t-1) - 3(t-3)\,u\,(t-3)$$

| By second shifting theorem

(iii) $$L^{-1}\left(\frac{p}{p^2-\omega^2}\right) = \cosh\omega t = F(t)$$

$$\therefore \qquad L^{-1}\left(\frac{pe^{-ap}}{p^2-\omega^2}\right) = \begin{matrix} \cosh\omega(t-a) & , \quad t>a \\ 0 & , \quad t<a \end{matrix}\Bigg\} = \cosh\omega(t-a)\,u\,(t-a)$$

| By second shifting theorem

3.10 Inverse Laplace Transform of Derivatives:

If $\qquad L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f^{(n)}(p) = L^{-1}\left[\frac{d^n}{dp^n}\{f(p)\}\right] = (-1)^n t^n F(t)$$

We have, $$L\{t^n F(t)\} = (-1)^n \left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n f^{(n)}(p)$$

$\therefore \qquad L^{-1}\{f^{(n)}(p)\} = (-1)^n t^n F(t)$

3.11 Multiplication by p

If $\qquad L^{-1}\{f(p)\} = F(t)$ and $F(0) = 0$, then $L^{-1}\{pf(p)\} = F'(t)$.

We have,

$$L\{F'(t)\} = pf(p) - F(0) = pf(p)$$

$\therefore \qquad L^{-1}\{p\,f(p)\} = F'(t)$

Note 1.If $\qquad F(0) \neq 0$, then

$$L^{-1}\{p\,f(p) - F(0)\} = F'(t) \quad \text{or} \quad L^{-1}\{pf(p)\} = F'(t) + F(0)\delta(t)$$

where $\delta(t)$ is the unit impulse function.

Note 2.Generalizations to $L^{-1}\{p^{(n)} f(p)\}$ are also possible for $n = 2, 3, \ldots.$

3.12 Division by p

If $\qquad L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(u)\,du$$

Also, $\qquad L^{-1}\left\{\frac{f(p)}{p^2}\right\} = \int_0^t \int_0^t F(u)\,du\,du$

$$L^{-1}\left\{\frac{f(p)}{p^3}\right\} = \int_0^t \int_0^t \int_0^t F(u)\,du\,du\,du$$

$\therefore \qquad L^{-1}\left\{\frac{f(p)}{p^n}\right\} = \int_0^t \int_0^t \ldots\ldots \int_{0\,(n\ times)}^t F(u)\,du \ldots\ldots \underset{(n\ times)}{du}$

3.13 Heaviside expansion formula for inverse Laplace transform

If $F(p)$ and $G(p)$ are two polynomials in p and the degree of $F(p)$ is less then the degree of $G(p)$ and if $G(p) = (p-\alpha_1)(p-\alpha_2)\ldots\ldots(p-\alpha_n)$

where $\alpha_1, \alpha_2 ......, \alpha_n$ are distinct constants, real or complex, then

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$$

By the method of partial frac $\frac{F(p)}{G(p)} = \frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + ... + \frac{A_r}{p-\alpha_r} + ... + \frac{A_n}{p-\alpha_n}$

Multiplying both sides by $p-\alpha_r$ and allowing $p \to \alpha_r$, we obtain,

$$A_r = \lim_{p\to\alpha_r} \frac{F(p)(p-\alpha_r)}{G(p)} = \lim_{p\to\alpha_r} F(p). \lim_{p\to\alpha_r} \frac{P-\alpha_r}{G(p)}$$

$$= \lim_{p\to\alpha_r} F(p). \lim_{p\to\alpha_r} \frac{1}{G'(p)} = \frac{F(\alpha_r)}{G'(\alpha_r)}$$

$$\therefore \quad \frac{F(p)}{G(p)} = \frac{F(\alpha_1)}{G'(\alpha_1)} . \frac{1}{p-\alpha_1} + ... + \frac{F(\alpha_r)}{G'(\alpha_r)} . \frac{1}{p-\alpha_r} + ... + \frac{F(\alpha_n)}{G'(\alpha_n)} . \frac{1}{p-\alpha_n}$$

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)} e^{\alpha_1 t} + ... + \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} + ... + \frac{F(\alpha_n)}{G'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} .$$

3.14 Convolution theorem:

If $\quad L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, then

$$L^{-1}\{f(p)g(p)\} = F * G = \int_0^1 F(u)G(t-u)\,du$$

Let $\quad \phi(t) = \int_0^1 F(u)G(t-u)\,du$ then

$$L\{\phi(t)\} = \int_0^\infty e^{-pt}\left[\int_0^t F(u)G(t-u)\,du\right]dt = \int_0^\infty \int_0^t e^{-pt} F(u)G(t-u)\,du\,dt$$

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-pt} F(u)G(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-pu} F(u) \left[\int_u^\infty e^{-p(t-u)} G(t-u)\, dt \right] du$$

$$= \int_0^\infty e^{-pu} F(u) \left[\int_0^\infty e^{-pv} G(v)\, dv \right] du$$

On putting $t - u = v$

$$= \int_0^\infty e^{-pu} F(u) g(p) du = g(p) \int_0^\infty e^{-pu} F(u)\, du$$

$$= g(p) . f(p) = f(p) g(p)$$

$\Rightarrow$ $$L^{-1}\{ f(p) g(p)\} = \phi(t) = \int_0^t F(u) G(t-u) du .$$

We call F* G, the convolution of F and G and the theorem is called the convolution theorem or the convolution property.

Q.8 Find the inverse Laplace transform of

(i) $\log\left(1 + \frac{1}{p^2}\right)$ (ii) $\log\left(\frac{p+1}{p-1}\right)$ (iii) $\cot^{-1}\left(\frac{p+3}{2}\right)$ (iv) $\tan^{-1}\left(\frac{2}{p^2}\right)$

(v) $\cot^{-1}\left(\frac{p}{2}\right)$ (vi) $L^{-1}\left[\log\left(\frac{p^2+4p+5}{p^2+2p+5}\right)\right]$

Sol. (i) Let $L^{-1}\left\{\log\left(1 + \frac{1}{p^2}\right)\right\} = F(t)$ | say

$\therefore$ $$L^{-1}\left[\frac{d}{dp}\left\{\log\left(1 + \frac{1}{p^2}\right)\right\}\right] = -t F(t)$$

$\Rightarrow$ $$L^{-1}\left[\frac{1}{1 + \frac{1}{p^2}}\left(\frac{-2}{p^3}\right)\right] = -t F(t)$$

$\Rightarrow \qquad L^{-1}\left[\dfrac{-2}{p(p^2+1)}\right] = -t\,F(t)$

$\Rightarrow \qquad L^{-1}\left[\dfrac{1}{p} - \dfrac{p}{p^2+1}\right] = \dfrac{t}{2}F(t)$

$\Rightarrow \qquad 1-\cos t = \dfrac{t}{2}F(t)$

$\therefore \qquad F(t) = \dfrac{2(1-\cos t)}{t}$

(ii) Let $\qquad L^{-1}\left\{\log\left(\dfrac{p+1}{p-1}\right)\right\} = F(t)$ | say

$\therefore \qquad L^{-1}\left[\dfrac{d}{dp}\{\log(p+1) - \log(p-1)\}\right] = -t\,F(t)$

$\Rightarrow \qquad L^{-1}\left[\dfrac{1}{p+1} - \dfrac{1}{p-1}\right] = -t\,F(t)$

$\Rightarrow \qquad e^{-t} - e^{t} = -t\,F(t)$

$\therefore \qquad F(t) = \dfrac{e^{t}-e^{-t}}{t} = \dfrac{2\sinh t}{t}$

(iii) Let $\qquad L^{-1}\left\{\cot^{-1}\left(\dfrac{p+3}{2}\right)\right\} = F(t)$

$\therefore \qquad L^{-1}\left[\dfrac{d}{dp}\left\{\cot^{-1}\left(\dfrac{p+3}{2}\right)\right\}\right] = -t\,F(t)$

$\Rightarrow \qquad L^{-1}\left[\dfrac{-1}{1+\left(\dfrac{p+3}{2}\right)^2}\cdot\dfrac{1}{2}\right] = -t\,F(t)$

$\Rightarrow \qquad L^{-1}\left[\dfrac{-2}{(p+3)^2+4}\right] = -t\,F(t)$

$\Rightarrow \qquad -e^{-3t}\sin 2t = -t\,F(t)$

$\Rightarrow \qquad F(t) = \dfrac{e^{-3t}\sin 2t}{t}$

(iv) Let $\qquad f(p) = \tan^{-1}\left(\dfrac{2}{p^2}\right) = \tan^{-1}\left(\dfrac{2}{1+p^2-1}\right)$

$$= \tan^{-1}\left\{\frac{2}{1+(p-1)(p+1)}\right\} = \tan^{-1}\frac{1}{p-1} - \tan^{-1}\frac{1}{p+1}$$

$\therefore \qquad L^{-1}\{f(p)\} = L^{-1}\left\{\tan^{-1}\left(\dfrac{1}{p-1}\right)\right\} - L^{-1}\left\{\tan^{-1}\left(\dfrac{1}{p+1}\right)\right\}$

$$= e^{t}L^{-1}\left(\tan^{-1}\frac{1}{p}\right) - = e^{-t}L^{-1}\left(\tan^{-1}\frac{1}{p}\right)$$

$$= 2\sinh t\, L^{-1}\left(\tan^{-1}\frac{1}{p}\right) = 2\sinh t\,.\frac{\sin t}{t}$$

$\Rightarrow \qquad F(t) = \dfrac{2}{t}\sin t\sinh t\,.$

(v) Let $\qquad L^{-1}\left\{\cot^{-1}\left(\dfrac{p}{2}\right)\right\} = F(t)$ $\qquad$ | say

$\therefore \qquad L^{-1}\left[\dfrac{d}{dp}\left\{\cot^{-1}\left(\dfrac{p}{2}\right)\right\}\right] = -t\,F(t)$

$\Rightarrow \qquad L^{-1}\left(\dfrac{-1}{1+\dfrac{p^2}{4}}\cdot\dfrac{1}{2}\right) = -t\,F(t)$

$\Rightarrow \qquad L^{-1}\left(\dfrac{-2}{p^2+4}\right) = -t\,F(t)$

$\Rightarrow \qquad -\sin 2t = -t\,F(t)$

$\therefore \qquad F(t) = \dfrac{\sin 2t}{t}$

Q.9 Find the inverse Laplace transform of

$$\frac{1}{p^4+4}.$$

Sol. $\quad p^4+4 = (p^2+2)^2 - (2p)^2 = (p^2-2p+2)(p^2+2p+2)$

$$\therefore \qquad \frac{1}{p^4+4} = \frac{1}{(p^2-2p+2)(p^2+2p+2)}$$

$$= \frac{1}{4p}\left[\frac{1}{p^2-2p+2} - \frac{1}{p^2+2p+2}\right]$$

Now, $L^{-1}\left(\dfrac{1}{p^2-2p+2}\right) = L^{-1}\left[\dfrac{1}{(p-1)^2+1}\right] = e^t \sin t$

and $L^{-1}\left(\dfrac{1}{p^2+2p+2}\right) = L^{-1}\left[\dfrac{1}{(p+1)^2+1}\right] = e^{-t} \sin t$

$$\therefore \ \frac{1}{4}L^{-1}\left(\frac{1}{p^2-2p+2} - \frac{1}{p^2+2p+2}\right) = \frac{1}{4}\left(e^t - e^{-t}\right)\sin t\,dt$$

Hence

$$L^{-1}\frac{1}{4p}\left[\left(\frac{1}{p^2-2p+2} - \frac{1}{p^2+2p+2}\right)\right] = \frac{1}{4}\int_0^t \left(e^t - e^{-t}\right)\sin t\,dt$$

$$\Rightarrow \qquad L^{-1}\left(\frac{1}{p^4+4}\right) = \frac{1}{4}\left[\frac{e^t}{2}(\sin t - \cos t) - \frac{e^{-t}}{2}(-\sin t - \cos t)\right]$$

$$= \frac{1}{4}\left[\sin t\left(\frac{e^t+e^{-t}}{2}\right) - \cos t\left(\frac{e^t+e^{-t}}{2}\right)\right]$$

or, $$L^{-1}\left(\frac{1}{p^4+4}\right)=\frac{1}{4}[\sin t\cosh t-\cos t\sinh t]$$

Q.10 Use convolution theorem to evaluate

(i) $L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$ (ii) $L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\}$.

Sol. (i) $$\frac{p}{(p^2+4)^2}=\frac{1}{p^2+4}.\frac{p}{p^2+4}$$

Let, $$f(p)=\frac{1}{p^2+4} \text{ and } g(p)=\frac{p}{p^2+4}$$

$\therefore$ $$F(t)=L^{-1}\{f(p)\}=L^{-1}\left(\frac{1}{p^2+4}\right)=\cos 2t$$

And $$G(t)=L^{-1}\{g(p)\}=L^{-1}\left(\frac{p}{p^2+4}\right)=\cos 2t$$

Now, $$F(u)=\frac{1}{2}\sin 2u,\ G(t-u)=\cos 2(t-u)$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}=\int_0^t \frac{1}{2}\sin 2u.\cos 2(t-u)du=\frac{1}{4}\int_0^t [\sin 2t+som(4u-2t)]du$$

$$=\frac{1}{4}\left[u\sin 2t-\frac{\cos(4u-2t)}{4}\right]_0^t=\frac{t}{4}\sin 2t$$

(ii) $$\frac{p^2}{(p^2+a^2)(p^2+b^2)}=\frac{p}{p^2+a^2}.\frac{p}{p^2+b^2}$$

Let, $$f(p)=\frac{p}{p^2+a^2} \text{ and } g(p)=\frac{p}{p^2+b^2}$$

$\therefore$ $$F(t)=L^{-1}\{f(p)\}=L^{-1}\left(\frac{p}{p^2+a^2}\right)=\cos at$$

and $$G(t) = L^{-1}\{g(p)\} = L^{-1}\left(\frac{p}{p^2+b^2}\right) = \cos bt$$

Now, $F(u) = \cos au$

$G(t-u) = \cos b(t-u)$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{p^2}{(p^2+a^2)(p^2+b^2)}\right\} = \int_0^t \cos au.\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos\{(a-b)u+bt\} + \cos\{(a+b)u-bt\}]\,du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u+bt\}}{a-b} + \frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt\}}{a+b}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

3.15 Applications to differential equations

Solution of ordinary linear differential equations with constant coefficients.

Laplace transforms can be used to solve ordinary linear differential equations with constant coefficients. The advantage of this method is that it yields the particular solution directly without the necessity of first finding the general solution and then evaluating the arbitrary constants.

Steps:

1. Take Laplace transform of both sides of the given differential equation, using initial conditions. This gives an algebraic equation.
2. Solve the algebraic equation to get $\bar{y}$ in terms of p.
3. Take Inverse Laplace transform of both sides. This gives y as a function of t which is the desired solution.

Remember: $L\{F^{(n)}(t)\} = p^n f(p) - p^{n-1}F(0) - p^{n-2}F'(0) - ...... - pF^{(n-2)}(0) - F^{(n-1)}(0)$ if $L\{F(t)\} = f(p)$

Q.9 Solve the differential equations

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0, \text{ where } y=1, \frac{dy}{dt}=2, \frac{d^2y}{dt^2}=2\, at\, t=0.$$

Sol. The given equation is $y'''+2y''-y'-2y=0$

Taking Laplace transform on both sides, we get

$$[p^3\bar{y}-p^2y(0)-py'(0)-y''(0)]+2[p^2\bar{y}-py(0)-y'(0)]-[p\bar{y}-y(0)]-2\bar{y}=0$$

Using the given conditions $y(0)=1, y'(0)=2, y''(0)=2,$ equation (1) reduces to

$$(p^3+2p^2-p-2)\bar{y}=p^2+4p+5$$

$$\therefore \qquad \bar{y}=\frac{p^2+4p+5}{p^3+2p^2-p-2}=\frac{p^2+4p+5}{(p-1)(p+1)(p+2)}$$

$$=\frac{5}{3(p-1)}-\frac{1}{p+1}+\frac{1}{3(p+2)} \qquad \text{(Partial Fractions)}$$

Taking the Inverse Laplace transforms of both sides, we get

$$y=\frac{5}{3}L^{-1}\left\{\frac{1}{p-1}\right\}-L^{-1}\left\{\frac{1}{p+1}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{p+2}\right\}=\frac{5}{3}e^t-e^{-t}+\frac{1}{3}e^{-2t}$$

or $$y=\frac{1}{3}(5e^t+e^{-2t})-e^{-t}.$$

Q.10 Using Laplace transform, find the solution of the initial value problem

$$\frac{d^2y}{dt^2}+9y=6\cos 3t$$

where $$y(0)=2, y'(0)=0$$

Sol. The given differential equation is

$$y''+9y=6\cos 3t$$

Taking Laplace transform on both sides of eqn. (1), we get

$$L(y'')+9L(y)=6L(\cos 3t)$$

$\Rightarrow \quad [p^2\bar{y} - py(0) - y'(0)] + 9\bar{y} = 6\dfrac{p}{p^2+9}$ $\quad$ | Here $\bar{y} = L(y)$

$\Rightarrow \quad (p^2+9)\bar{y} - 2p = \dfrac{6p}{p^2+9}$ $\quad$ $|\because\ y(0)=2, y'(0)=0$

$\Rightarrow \quad \bar{y} = \dfrac{6p}{(p^2+9)^2} + \dfrac{2p}{p^2+9}$

Taking Inverse Laplace transform on both sides of (2), we get

$$y(t) = t\sin 3t + 2\cos 3t \qquad \left|\because\ L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{t}{2a}\sin at\right.$$

3.16 Solution of simultaneous ordinary differential equations.

Laplace transform technique can also be used in solving two or more simultaneous ordinary differential equations.

This procedure is illustrated as follows

Q.11 Solve the simultaneous equations

$$\frac{dx}{dt} - y = e^t, \frac{dy}{dt} + x = \sin t, \text{ given } x(0)=1,\ y(0)=0.$$

Sol. Taking Laplace transform of the given equations, we get

$$[p\bar{x} - x(0)] - \bar{y} = \frac{1}{p-1}$$

i.e., $\quad p\bar{x} - 1 - \bar{y} = \dfrac{1}{p-1}$ $\quad$ $[\because\ x(0)=1]$

i.e., $\quad p\bar{x} - \bar{y} = \dfrac{1}{p-1}$ $\quad$ …(1)

and $\quad [p\bar{y} - y(0)] + \bar{x} = \dfrac{1}{p^2-1}$

i.e., $\bar{x} + p\bar{y} = \frac{1}{p^2-1}$ …(2) $[\because \ y(0)=0]$

Solving (1) and (2) for $\bar{x}$ and $\bar{y}$, we have

$$\bar{x} = \frac{p^2}{(p-1)(p^2+1)} + \frac{1}{(p^2+1)^2} = \frac{1}{2}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + \frac{1}{(p^2+1)^2}$$

and $$\bar{y} = \frac{p}{(p^2+1)^2} - \frac{p}{(p-1)(p^2+1)^2} = \frac{p}{(p^2+1)^2} = \frac{1}{2}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$$

Taking Inverse Laplace transform of both sides, we get

$$x = \frac{1}{2}L^{-1}\left[\frac{1}{p-1} + \frac{p}{p^2+1} + \frac{1}{p^2+1}\right] + L^{-1}\left[\frac{1}{(p^2+1)^2}\right]$$

$$= \frac{1}{2}[e^t + \cos t + \sin t] + \frac{1}{2}(\sin t - t\cos t)$$

$$\left[\because \ L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)\right]$$

$$= \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$$

$$y = L^{-1}\left[\frac{p}{(p^2+1)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{p-1} - \frac{p}{p^2+1} + \frac{1}{p^2+1}\right]$$

$$= \frac{1}{2}t\sin t - \frac{1}{2}[e^t - \cos t + \sin t] \qquad \left[\because \ L^{-1}\left[\frac{p}{(p^2+a^2)^2}\right] = \frac{1}{2a}t\sin at\right]$$

$$= \frac{1}{2}[t\sin t - e^t + \cos t - \sin t]$$

Hence $x = \frac{1}{2}(e^t + \cos t + 2\sin t - t\cos t)$

$$y = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t).$$

Q.12 Solve the integral equation

$$y(t) = t^2 + \int_0^t y(u).\sin(t-u)\,du.$$

Sol. We have

$$y(t) = t^2 + y(t) * \sin t$$

Let $L\{y(t)\} = \bar{y}(p)$ then taking Laplace transform and using convolution theorem, we find that

$$\bar{y} = \frac{2}{p^3} + \bar{y}.\frac{1}{p^2+1}$$

$$\Rightarrow \quad \bar{y}\left(1 - \frac{1}{p^2+1}\right) = \frac{2}{p^3} \quad \Rightarrow \quad \bar{y}\left(1 - \frac{p^2}{p^2+1}\right) = \frac{2}{p^3}$$

$$\Rightarrow \quad \bar{y} = \frac{2(p^2+1)}{p^5} = \frac{2}{p^3} + \frac{2}{p^5}$$

Taking Inverse Laplace transform, we get

$$y = t^2 + \frac{t^4}{12}.$$

Q. A function $f(t)$ obeys the equation

$f(t) + 2\int_0^t f(t)dt = \cosh 2t$ Find the Laplace transform of $f(t)$.

Sol. Taking the Laplace transform of the given equation,

We have $\quad L\{f(t)\} + 2L\left(\int_0^t f(t)dt\right) = L(\cosh 2t)$

$$\Rightarrow \quad \bar{f}(p) + \frac{2}{p}\bar{f}(p) = \frac{p}{p^2-4} \text{ where } \bar{f}(p) = L\{f(t)\}$$

$$\Rightarrow \qquad \left(\frac{p+2}{p}\right)\overline{f}(p) = \frac{p}{p^2-4}$$

$$\Rightarrow \qquad \overline{f}(p) = \frac{p^2}{(p+2)(p^2-4)} = \frac{p^2}{(p+2)^2(p-2)}$$

or, $$L\{f(t)\} = \frac{p^2}{(p+2)^2(p-2)}$$

which is the required result

Assignment (Unit-I) (Laplace Transform)

(1) Evaluate the Laplace Transform of

(a) $e^{2t} + 4t^3 - 2\sin 3t$ **(b)** $\sin 2t \sin 3t$ **(c)** $\sin^2 3t$ **(d)** $e^{-at}\sinh bt$

(e) $f(t) = \begin{cases} \sin t\,, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$ **(f)** $t\sin^2 t$

(g) $t^2 \cos at$ (h) $te^{-2t}\sin 2t$ (i) $\dfrac{\sin at}{t}$ (j) $\dfrac{e^{-t}\sin t}{t}$ (k) $\dfrac{1-\cos 2t}{t}$

(l) $\int_0^\infty te^{-2t}\cos t\,dt$ **(m)** $\int_0^\infty \dfrac{\sin mt}{t}\,dt$.

(2) Evaluate the Inverse Laplace Transform of

(a) $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ **(b)** $\dfrac{4s+5}{(s-1)^2(s+2)}$ **(c)** $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$ **(d)** $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

(e) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ **(f)** $\dfrac{2s-1}{s^2+4s+13}$ **(g)** $\dfrac{s^2}{(s^2+a^2)^2}$ **(h)** $\dfrac{1}{(s^2+a^2)^2}$

(3) Find Inverse Laplace Transform, Using Convolution Theorem

(a) $\dfrac{1}{(s+a)(s+b)}$ **(b)** $\dfrac{1}{s^2(s+1)^2}$ **(c)** $\dfrac{1}{(s-2)(s+2)^2}$ **(d)** $\dfrac{s}{(s^2+1)(s^2+4)}$

(4) Solve the equations by Laplace transform method

(a) $\frac{dx}{dt} = x = \sin\omega t$ **(b)** $y'' - 3y' + 2y = 4t + e^{3t}$,y(0) = 1, $y'(0) = -1$ **(c)** $(D^2+1)x = t\cos 2t$, x = Dx = 0 at t = 0. **(d)** $y'' + 2y' + 5y = e^t \sin t$, y (0) = 0, $y'(0) = 1$ **(e)** $ty'' + 2y' + ty = \cos t$, y(0) = 1.

(5) Solve the simultaneous equation , by using Laplace Transform

(a) $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given x(0) = 1, y(0) = 0 .

(b) $D^2x + 3x - 2y = 0, D^2x + D^2y - 3x + 5y = 0$, where $D = \frac{d}{dt}$, If x = 0, y = 0, Dx = 3, Dy = 2 when t=0

(6) Find Laplace Transform of

(a) f(t) = $\begin{cases} t-1, 1<t<2 \\ 3-t, 2<t<3 \end{cases}$ **(b)** f(t) = $\begin{cases} 2t, & 0<t<\pi \\ 1, & t>\pi \end{cases}$ **(c)** $\frac{e^{-\pi s}}{s^2+1}$ **(d)** $\frac{e^{-s}}{(s-1)(s-2)}$

(7) Find Laplace Transform of

(a) f(t) = $\begin{cases} \sin\omega t, & 0<t<\frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases}$ **(b)** f(t) = $\begin{cases} 1, & 0<t<\frac{a}{2} \\ -1, & \frac{a}{2}<t<a \end{cases}$ **(c)** $e^{-at}J_{\circ}(bt)$

(8) Find the inverse Laplace Transform of $f(s) = \log\frac{s+a}{s+b}$.

(9) Find $L^{-1}\left\{\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right\}$

(10) Evaluate $L^{-1}\left[\frac{e^{-s} - 3e^{-3s}}{s^2}\right]$

(11) Solve, by the method of Laplace transform, the differential equation $(D^2+n^2)x = a\sin(nt+\alpha)$, x = Dx = 0 at t = 0.

(12) Using Laplace Transform, Solve the following differential equation :
$y'' + 2ty' - y = t,\ when\ y(0) = 0\ and\ y'(0) = 1$.